THE F-PURE THRESHOLD OF A DETERMINANTAL IDEAL

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ABSTRACT. The F-pure threshold is a numerical invariant of prime characteristic singularities, that constitutes an analogue of the log canonical thresholds in characteristic zero. We compute the F-pure thresholds of determinantal ideals, i.e., of ideals generated by the minors of a generic matrix.

1. INTRODUCTION

Consider the ring of polynomials in a matrix of indeterminates X, with coefficients in a field of prime characteristic. We compute the F-pure thresholds of determinantal ideals, i.e., of ideals generated by the minors of X of a fixed size.

The notion of *F-pure thresholds* is due to Takagi and Watanabe [TW], see also Mustaţă, Takagi, and Watanabe [MTW]. These are positive characteristic invariants of singularities, analogous to log canonical thresholds in characteristic zero. While the definition exists in greater generality—see the above papers—the following is adequate for our purpose:

Definition 1.1. Let R be a polynomial ring over a field of characteristic $p > 0$, with the homogeneous maximal ideal denoted by $\mathfrak{m}$. For a homogeneous proper ideal I, and integer $q = p^e$, set

$$\nu_I(q) = \max\left\{r \in \mathbb{N} \mid I^r \not\subseteq \mathfrak{m}^{[q]}\right\},$$

where $\mathfrak{m}^{[q]} = (a^q \mid a \in \mathfrak{m})$. If I is generated by N elements, it is readily seen that $0 \leqslant \nu_I(q) \leqslant N(q-1)$. Moreover, if $f \in I^r \setminus \mathfrak{m}^{[q]}$, then $f^p \in I^{pr} \setminus \mathfrak{m}^{[pq]}$. Thus,

$$\nu_I(pq) \geqslant p\nu_I(q).$$

It follows that $\left\{\nu_I(p^e)/p^e\right\}_{e\geqslant 1}$ is a bounded monotone sequence; its limit is the *F-pure threshold of I*, denoted $\operatorname{fpt}(I)$.

The F-pure threshold is known to be rational in a number of cases, see, for example, [BMS1, BMS2, BSTZ, Ha, KLZ]. The theory of F-pure thresholds is motivated by connections to log canonical thresholds; for simplicity, and to conform to the above context, let I be a homogeneous ideal in a polynomial ring over the field of rational numbers. Using I_p for the corresponding characteristic p models, one has the inequality

$$\operatorname{fpt}(I_p) \leqslant \operatorname{lct}(I) \qquad \text{for all } p \gg 0,$$

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where $\operatorname{lct}(I)$ denotes the log canonical threshold of I. Moreover,

$$\lim_{p\longrightarrow\infty} \operatorname{fpt}(I_p) = \operatorname{lct}(I)\,.$$

These statements follow from the work of Hara and Yoshida [HY], and may be found as [MTW, Theorems 3.3, 3.4].

The F-pure thresholds of defining ideals of elliptic curves in $\mathbb{P}^2$ are computed by Bhatt [Bh]; this is extended to Calabi-Yau hypersurfaces in [BS]. Hernández has computed F-pure thresholds for binomial hypersurfaces [He1] and for diagonal hypersurfaces [He2]. In the present paper, we perform the computation for determinantal ideals:

Theorem 1.2. *Fix positive integers $t \leqslant m \leqslant n$, and let X be an $m \times n$ matrix of indeterminates over a field $\mathbb{F}$ of prime characteristic. Let R be the polynomial ring $\mathbb{F}[X]$, and I_t the ideal generated by the size t minors of X.*

The F-pure threshold of I_t is

$$\min\left\{\frac{(m-k)(n-k)}{t-k} \mid k = 0,\dots,t-1\right\}.$$

It follows that the F-pure threshold of a determinantal ideal is independent of the characteristic. For each prime characteristic, it agrees with the log canonical threshold of the corresponding characteristic zero determinantal ideal, as computed by Johnson [Jo, Theorem 6.1], or Docampo [Do, Theorem 5.6]. It is indeed possible to use resolutions of determinantal ideals as in [Va], and the characteristic zero calculations of log canonical threshold, to obtain bounds on the F-pure threshold in each prime characteristic; however, we have opted to perform all calculations in this paper entirely in the prime characteristic setting.

2. The computations

The primary decomposition of powers of determinantal ideals, i.e., of the ideals I_t^s, was computed by DeConcini, Eisenbud, and Procesi [DEP] in the case of characteristic zero, and extended to the case of *non-exceptional* prime characteristic by Bruns and Vetter [BV, Chapter 10]. By Bruns [Br, Theorem 1.3], the intersection of the primary ideals arising in a primary decomposition of I_t^s in non-exceptional characteristics, yields, in all characteristics, the integral closure $\overline{I_t^s}$. We record this below in the form that is used later in the paper:

Theorem 2.1 (Bruns)**.** *Let s be a positive integer, and let $\delta_1,\dots,\delta_h$ be minors of the matrix X. If $h \leqslant s$ and $\sum_i \deg\delta_i = ts$, then*

$$\delta_1\cdots\delta_h \ \in\ \overline{I_t^s}\,.$$

Proof. By [Br, Theorem 1.3], the ideal $\overline{I_t^s}$ has a primary decomposition

$$\bigcap_{j=1}^{t} I_j^{((t-j+1)s)}\,.$$

Thus, it suffices to verify that

$$\delta_1\cdots\delta_h \ \in\ I_j^{((t-j+1)s)}$$

for each j with $1 \leqslant j \leqslant t$. This follows from [BV, Theorem 10.4]. $\square$

We will also need:

Lemma 2.2. *Let k be the least integer in the interval $[0, t-1]$ such that*

$$\frac{(m-k)(n-k)}{t-k} \leqslant \frac{(m-k-1)(n-k-1)}{t-k-1};$$

interpreting a positive integer divided by zero as infinity, such a k indeed exists. Set

$$u = t(m+n-2k) - mn + k^2.$$

Then $t-k-u \geqslant 0$. Moreover, if k is nonzero, then $t-k+u > 0$.

Proof. Rearranging the inequality above, we have

$$t(m+n-2k-1) \leqslant mn - k^2 - k,$$

which gives $t-k-u \geqslant 0$. If k is nonzero, then the minimality of k implies that

$$t(m+n-2k+1) > mn - k^2 + k,$$

equivalently, that $t-k+u > 0$. $\square$

Proof of Theorem 1.2. We first show that for each k with $0 \leqslant k \leqslant t-1$, one has

$$\operatorname{fpt}(I_t) \leqslant \frac{(m-k)(n-k)}{t-k}.$$

Let δ_k and δ_t be minors of size k and t respectively. Theorem 2.1 implies that

$$\delta_k^{t-k-1}\delta_t \in \overline{I_{k+1}^{t-k}},$$

and hence that $\delta_k^{t-k-1} I_t \subseteq \overline{I_{k+1}^{t-k}}$. By the Briançon-Skoda theorem, see, for example, [HH, Theorem 5.4], there exists an integer N such that

$$\left(\delta_k^{t-k-1} I_t\right)^{N+l} \in I_{k+1}^{(t-k)l}$$

for each integer $l \geqslant 1$. Localizing at the prime ideal I_{k+1} of R, one has

$$I_t^{N+l} \subseteq I_{k+1}^{(t-k)l} R_{I_{k+1}} \qquad \text{for each } l \geqslant 1,$$

as the element δ_k is a unit in $R_{I_{k+1}}$. Since $R_{I_{k+1}}$ is a regular local ring of dimension $(m-k)(n-k)$, with maximal ideal $I_{k+1}R_{I_{k+1}}$, it follows that

$$I_t^{N+l} \subseteq I_{k+1}^{[q]} R_{I_{k+1}}$$

for positive integers l and $q = p^e$ satisfying

$$(t-k)l > (q-1)(m-k)(n-k).$$

Returning to the polynomial ring R, the ideal I_{k+1} is the unique associated prime of $I_{k+1}^{[q]}$; this follows from the flatness of the Frobenius endomorphism, see for example, [ILL, Corollary 21.11]. Hence, in the ring R, we have

$$I_t^{N+l} \subseteq I_{k+1}^{[q]}$$

for all integers q, l satisfying the above inequality. This implies that

$$\nu_{I_t}(q) \leqslant N+1+\frac{(q-1)(m-k)(n-k)}{t-k}.$$

Dividing by q and passing to the limit, one obtains

$$\operatorname{fpt}(I_t) \leqslant \frac{(m-k)(n-k)}{t-k}.$$

Next, fix k and u be as in Lemma 2.2. Set Δ_k to be the product of minors:

$$\prod_{i=1}^{n-m+1}[1,\dots,m \mid i,\dots,i+m-1]$$
$$\times\prod_{j=2}^{m-k}[j,\dots,m \mid 1,\dots,m-j+1]\cdot[1,\dots,m-j+1 \mid n-m+j,\dots,n].$$

If $k>0$, we set Δ'_k to be

$$\Delta_k\cdot[m-k+1,\dots,m \mid 1,\dots,k].$$

Notice that $\deg\Delta_k = mn-k^2-k$ and that $\deg\Delta'_k = mn-k^2$. The element Δ_k is a product of $m+n-2k-1$ minors and Δ'_k of $m+n-2k$ minors.

Finally, set

$$\Delta = \begin{cases} \Delta_0^t & \text{if} \quad k=0,\\ \Delta_k^u\cdot(\Delta'_k)^{t-k-u} & \text{if} \quad k\geqslant 1 \text{ and } u\geqslant 0,\\ (\Delta'_k)^{t-k+u}\cdot\Delta_{k-1}^{-u} & \text{if} \quad k\geqslant 1 \text{ and } u<0. \end{cases}$$

We claim that Δ belongs to the integral closure of the ideal $I_t^{(m-k)(n-k)}$. This holds by Theorem 2.1, since, in each case,

$$\deg\Delta = t(m-k)(n-k),$$

and Δ is a product of at most $(m-k)(n-k)$ minors: if $k\geqslant 1$, then Δ is a product of exactly $(m-k)(n-k)$ minors, whereas if $k=0$ then Δ is a product of $t(m+n-1)$ minors and

$$t(m+n-1) \leqslant mn.$$

Let $\mathfrak{m}$ be the homogeneous maximal ideal of R. For a positive integer s that is not necessarily a power of p, set

$$\mathfrak{m}^{[s]} = (x_{ij}^s \mid i=1,\dots,m,\ j=1,\dots,n).$$

Using the lexicographical term order on R where

$$x_{11}>x_{12}>\cdots>x_{1n}>x_{21}>\cdots>x_{m1}>\cdots>x_{mn},$$

the initial forms $\operatorname{in}(\Delta_k)$ and $\operatorname{in}(\Delta'_k)$ are square-free monomials, and

$$\operatorname{in}(\Delta) = \begin{cases} \operatorname{in}(\Delta_0)^t & \text{if} \quad k=0,\\ \operatorname{in}(\Delta_k)^u\cdot\operatorname{in}(\Delta'_k)^{t-k-u} & \text{if} \quad k\geqslant 1 \text{ and } u\geqslant 0,\\ \operatorname{in}(\Delta'_k)^{t-k+u}\cdot\operatorname{in}(\Delta_{k-1})^{-u} & \text{if} \quad k\geqslant 1 \text{ and } u<0. \end{cases}$$

Thus, each variable x_{ij} occurs in the monomial $\operatorname{in}(\Delta)$ with exponent at most $t-k$. It follows that

$$\Delta \notin \mathfrak{m}^{[t-k+1]}.$$

As Δ belongs to the integral closure of $I_t^{(m-k)(n-k)}$, there exists a nonzero homogeneous polynomial $f \in R$ such that

$$f\Delta^l \in I_t^{(m-k)(n-k)l} \qquad \text{for all integers } l \geqslant 1\,.$$

But then

$$f\Delta^l \in I_t^{(m-k)(n-k)l} \setminus \mathfrak{m}^{[q]}$$

for all integers l with $\deg f + l(t-k) \leqslant q-1$. Hence,

$$\nu_{I_t}(q) \geqslant (m-k)(n-k)l \qquad \text{for all integers } l \text{ with } l \leqslant \frac{q-1-\deg f}{t-k}\,.$$

Thus,

$$\nu_{I_t}(q) \geqslant (m-k)(n-k)\left(\frac{q-1-\deg f}{t-k}-1\right),$$

and passing to the limit, one obtains

$$\operatorname{fpt}(I_t) \geqslant \frac{(m-k)(n-k)}{t-k},$$

which completes the proof. □

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